INSTABILITIES IN FLUIDS

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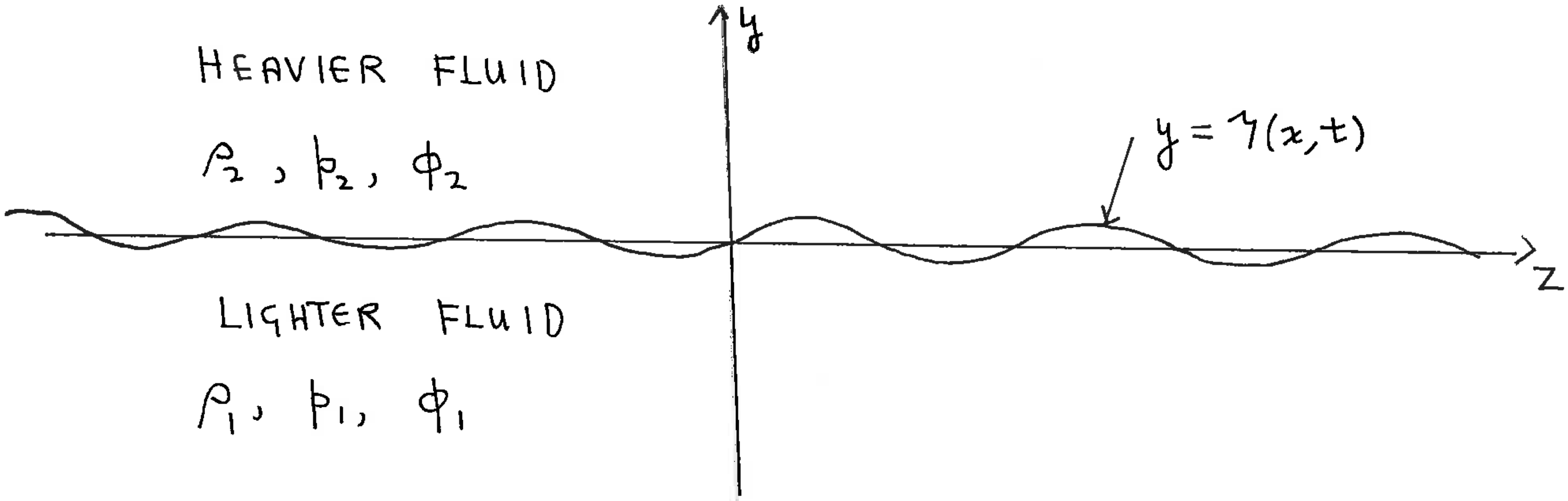
- GRADUATE MODELLING CAMP 2018

1. RAYLEIGH - TAYLOR INSTABILITY



IRROTATIONAL $\quad \underline{\omega} = \underline{0}, \qquad \underline{V} = \underline{\nabla}\phi$

INVISCID $\quad \mu = 0$

INCOMPRESSIBLE $\quad \underline{\nabla} \cdot \underline{V} = 0$

$$\underline{\nabla} \cdot \underline{\nabla}\phi = 0, \qquad \nabla^2 \phi = 0$$

LINEARISE: NEGLECT PRODUCTS AND SQUARES

EQUATIONS:

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0$$

$$\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} = 0$$

INTERFACIAL CONDITIONS:

$y = 0$ $$\frac{\partial \phi_1}{\partial y}(x,0,t) = \frac{\partial \eta}{\partial t}(x,t)$$

$y = 0$ $$\frac{\partial \phi_2}{\partial y}(x,0,t) = \frac{\partial \eta}{\partial t}(x,t)$$

$y = 0$ $$p_1(x,0,t) = p_2(x,0,t)$$

$$\rho\left[\frac{\partial \phi_1}{\partial t}(x,0,t) + g\,\eta(x,t)\right] = \rho_2\left[\frac{\partial \phi_2}{\partial t}(x,0,t) + g\,\eta(x,t)\right]$$

BERNOULLI'S EQUATION FOR $\underline{\omega} = \underline{0}$

LOOK FOR SOLUTION OF FORM

$$\phi_1(x,y,t) = F_1(y)\exp[i(kx-\omega t)]$$

$$\phi_2(x,y,t) = F_2(y)\exp[i(kx-\omega t)]$$

$k = \frac{2\pi}{\lambda}$ = WAVE NUMBER

λ = WAVE LENGTH

$\omega = \frac{2\pi}{T}$ = ANGULAR FREQUENCY

$\frac{1}{T}$ = FREQUENCY OF WAVE

T = PERIOD OF WAVE

$F(y)$ = AMPLITUDE

$\frac{\omega}{k} = \frac{\lambda}{T} = c$ = PHASE VELOCITY OF WAVE

DERIVE A <u>DISPERSION RELATION</u>

$$\omega = \omega(k)$$

FIND

$$\omega = \alpha(k) + i\,\beta(k)$$

DISTURBANCE AT INTERFACE

$$\eta(x,t) = \eta_0 \exp\left[\, i(kx - \omega t)\right]$$

$$= \eta_0 \exp\left[\, i(kx - (\alpha + i\beta)t)\right]$$

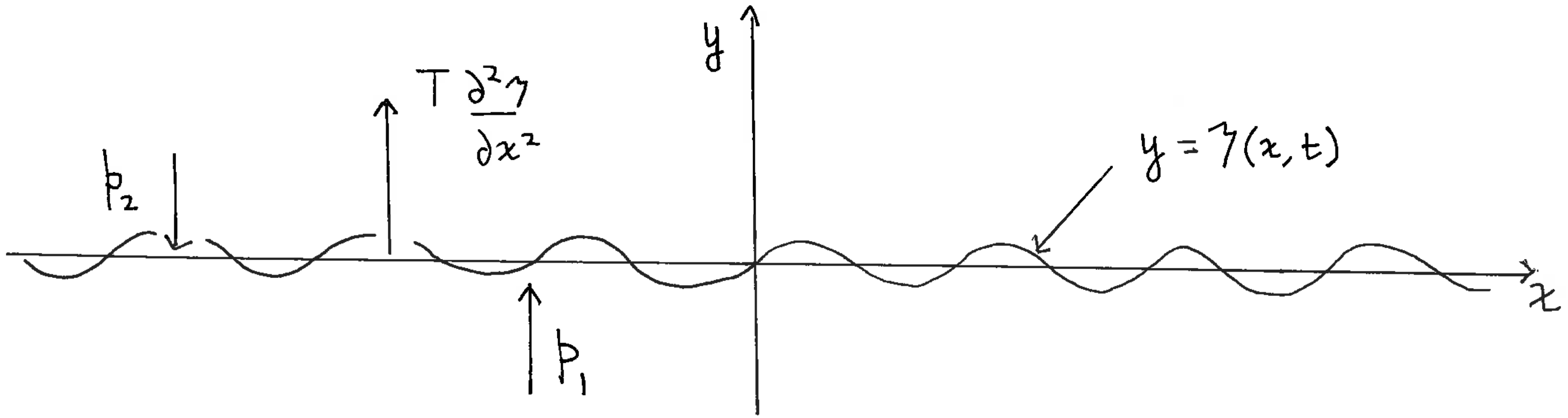
$$= \eta_0 \exp(\beta t)\, \exp\left[\, i(kx - \alpha t)\right]$$

$$= \eta_0 \exp(\beta t)\left[\cos(kx - \alpha t) + i \sin(kx - \alpha t)\right]$$

$\beta < 0$ INTERFACE STABLE

$\beta > 0$ INTERFACE UNSTABLE

2 EFFECT OF INTERFACIAL TENSION ON RAYLEIGH-TAYLOR INSTABILITY



NET UPWARD FORCE PER UNIT AREA DUE TO INTERFACIAL TENSION

$$= T\frac{\partial^2\eta}{\partial x^2}$$

$$p_1(x,0,t) + T\frac{\partial^2\eta}{\partial x^2}(x,t) = p_2(x,0,t)$$

DERIVE DISPERSION RELATION

$$\omega = \omega(k)$$

FIND λ_c SUCH THAT

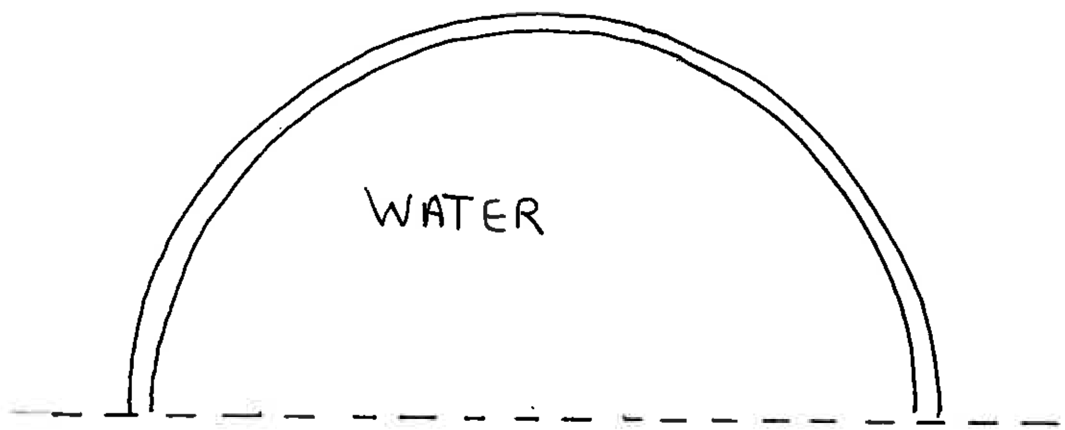
STABLE $\quad \lambda < \lambda_c(T, \rho_1, \rho_2, g)$

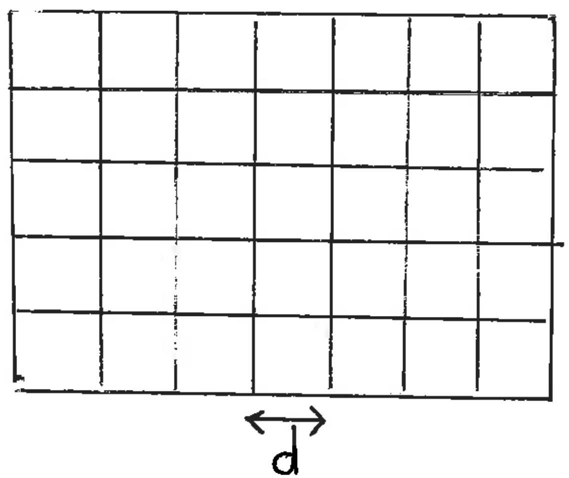
UNSTABLE $\quad \lambda > \lambda_c(T, \rho_1, \rho_2, g)$

SURFACE TENSION STABILISES INTERFACE AGAINST SHORT WAVELENGTH DISTURBANCES

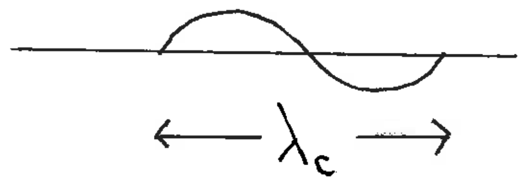
- APPLICATION

WATER CAN BE RETAINED IN AN INVERTED GLASS IF MOUTH OF GLASS IS COVERED BY A FINE-MESHED GAUZE

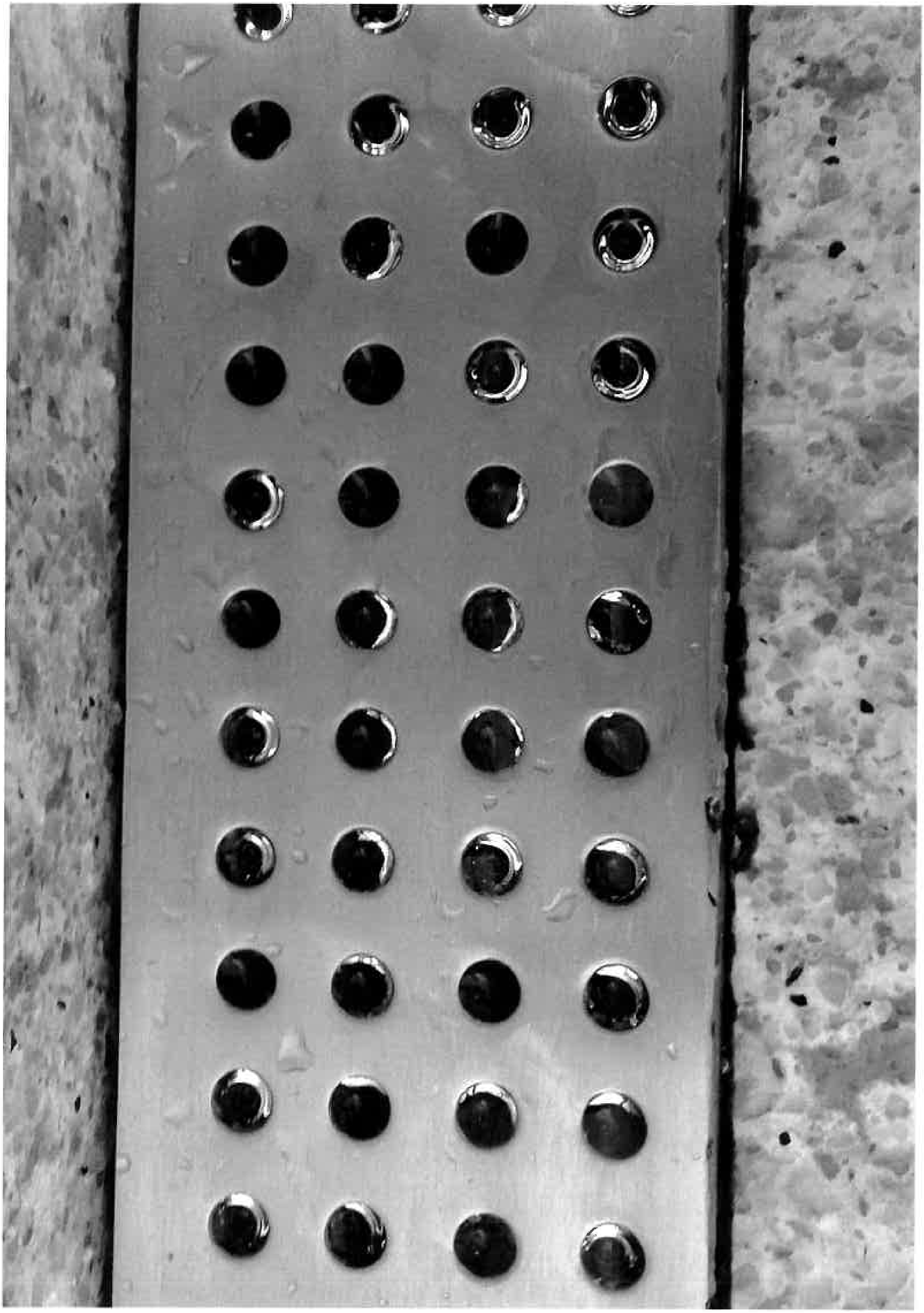




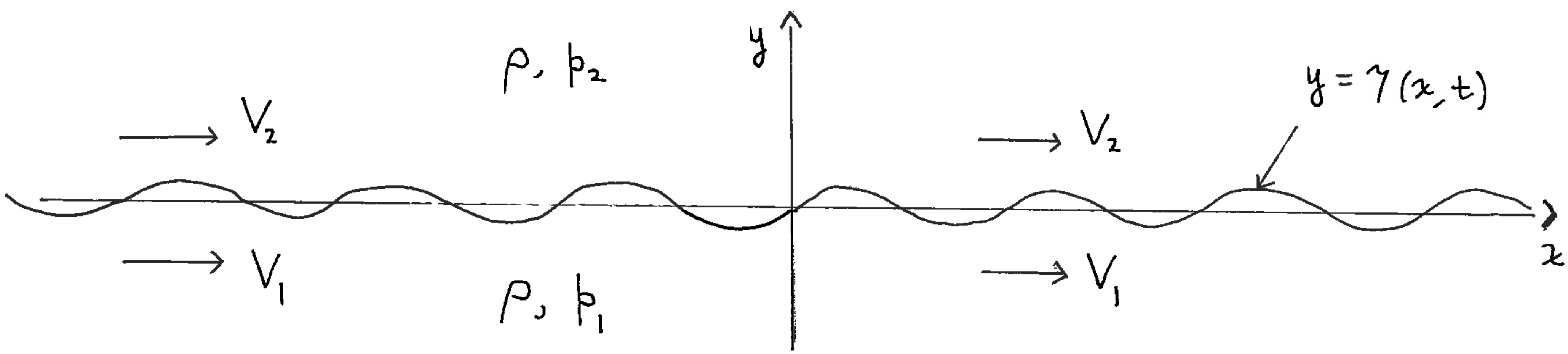
FOR STABILITY $d < \frac{\lambda_c}{2}$



IF THE SPACING IN THE WIRE GAUZE IS GREATER THAN $\frac{\lambda_c}{2}$ WATER CANNOT BE RETAINED IN THE INVERTED GLASS



3 KELVIN - HELMHOLTZ INSTABILITY



IRROTATIONAL, INVISCID, INCOMPRESSIBLE

FLUID DIVIDED BY A THIN MEMBRANE (FLAG OR SAIL OF YACHT)

FLUID FLOWING WITH CONSTANT VELOCITY $\begin{cases} V_2 \text{ IN UPPER REGION} \\ V_1 \text{ IN LOWER REGION} \end{cases}$

MEMBRANE PERTURBED BY SMALL DISTURBANCE

$$y = \eta(x,t)$$

- FLUID VELOCITIES

REGION 1 $\qquad V_x^{(1)} = V_1 + \dfrac{\partial \phi_1}{\partial x}, \qquad V_y^{(1)} = \dfrac{\partial \phi_1}{\partial y},$

REGION 2 $\qquad V_x^{(2)} = V_2 + \dfrac{\partial \phi_2}{\partial x}, \qquad V_y^{(2)} = \dfrac{\partial \phi_2}{\partial y}.$

NEGLECT POWERS AND PRODUCTS OF SMALL QUANTITIES

- EQUATIONS:

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0$$

$$\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} = 0$$

- INTERFACIAL CONDITIONS:

$$y = 0: \quad \frac{\partial \phi_1}{\partial y}(x,0,t) = \frac{\partial \eta}{\partial t}(x,t) + V_1 \frac{\partial \eta}{\partial x}(x,t) \qquad \left(= \frac{D\eta}{Dt}\right)$$

$$y = 0: \quad \frac{\partial \phi_2}{\partial y}(x,0,t) = \frac{\partial \eta}{\partial t}(x,t) + V_2 \frac{\partial \eta}{\partial x}(x,t)$$

$$y = 0: \quad p_1(x,0,t) = p_2(x,0,t)$$

$$\frac{\partial \phi_1}{\partial t}(x,0,t) + V_1 \frac{\partial \phi_1}{\partial x}(x,0,t) = \frac{\partial \phi_2}{\partial t}(x,0,t) + V_2 \frac{\partial \phi_2}{\partial x}(x,0,t)$$

BERNOULLI'S EQUATION FOR $\omega = 0$

LOOK FOR SOLUTION OF FORM

$$\phi_1(x,y,t) = F_1(y)\exp\left[i(kx-\omega t)\right]$$

$$\phi_2(x,y,t) = F_2(y)\exp\left[i(kx-\omega t)\right]$$

$$\eta(x,t) = \eta_0\exp\left[i(kx-\omega t)\right]$$

DERIVE DISPERSION RELATION

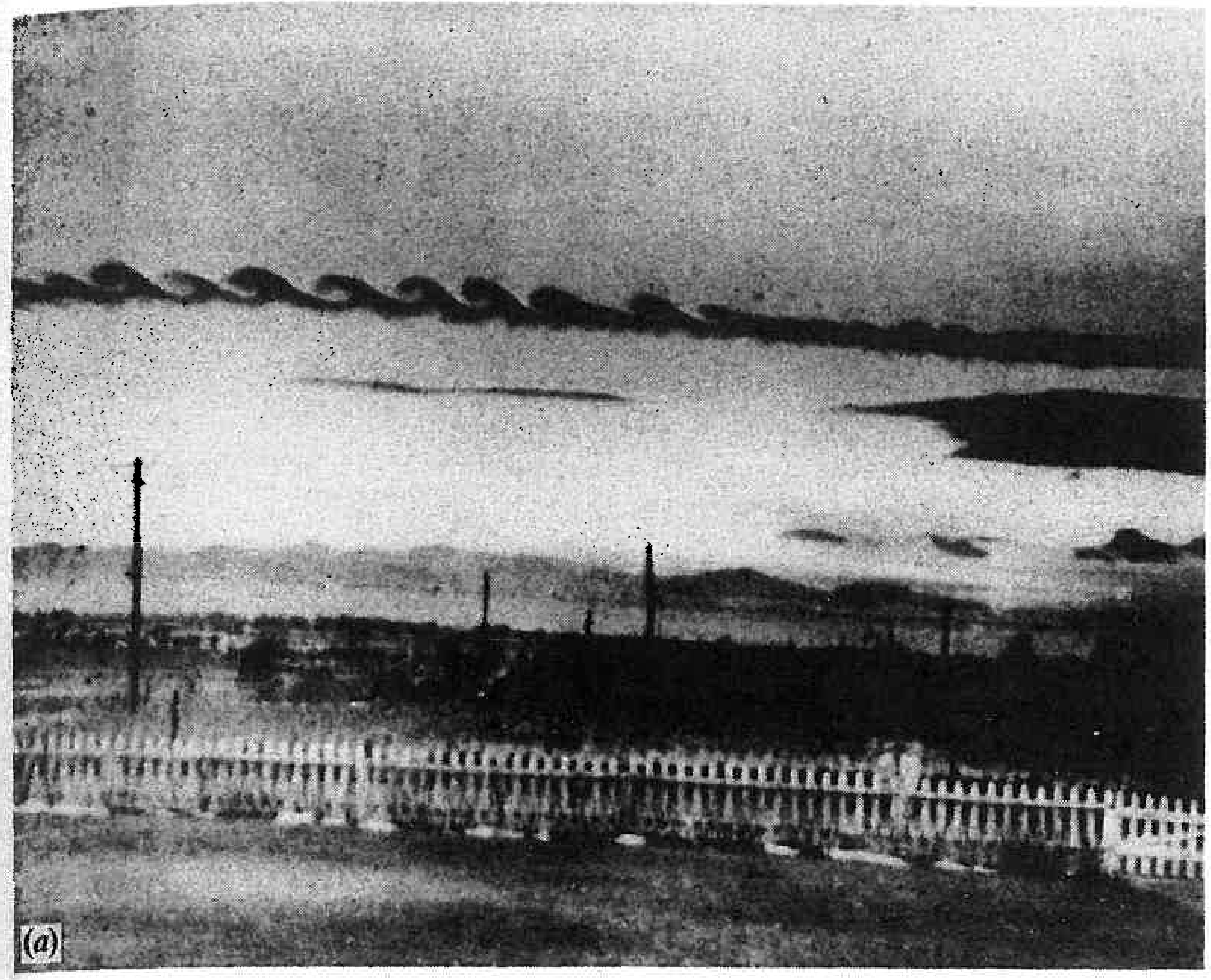
$$\omega = \omega(k)$$

INVESTIGATE STABILITY OF FLOW

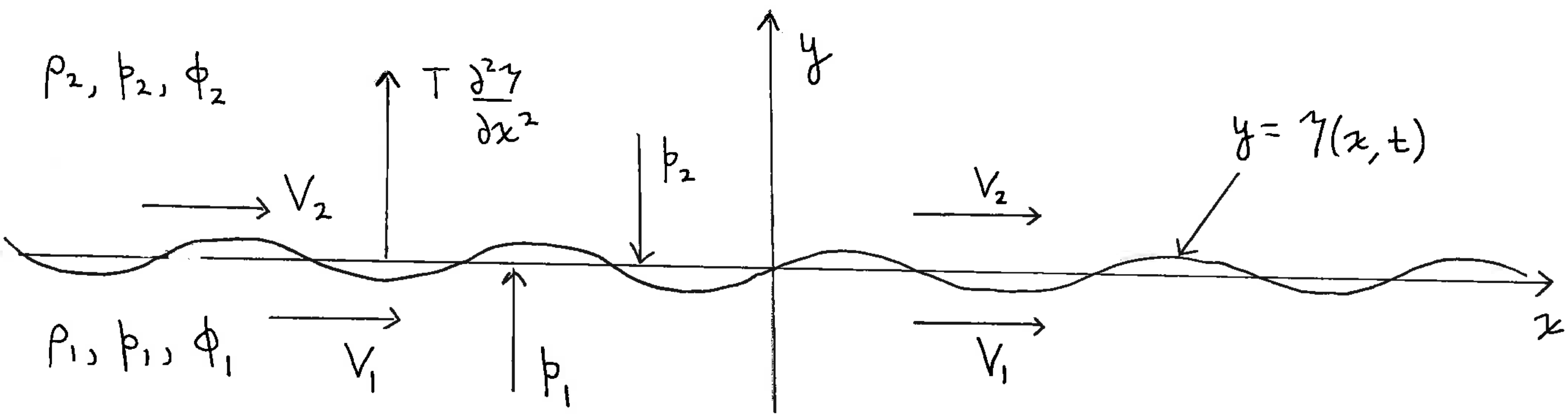
EXPLAINS

- FLAPPING OF FLAGS IN THE WIND
- FLAPPING OF SAILS OF A YACHT IN THE WIND
- CLEAR-AIR TURBULENCE IN ATMOSPHERE

KELVIN-HELMHOLTZ INSTABILITY

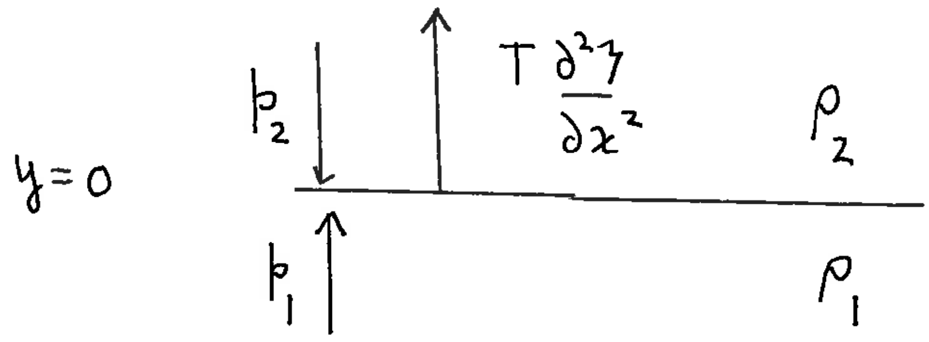


4 EFFECT OF GRAVITY AND INTERFACIAL TENSION ON KELVIN-HELMHOLTZ INSTABILITY



NET UPWARD FORCE PER UNIT AREA DUE TO INTERFACIAL TENSION

$$= T \frac{\partial^2 \eta}{\partial x^2}$$



$$y=0: \quad p_1(x,0,t) + T\frac{\partial^2\eta}{\partial x^2} = p_2(x,0,t)$$

$$y=0: \quad \rho_1\left[\frac{\partial\phi_1}{\partial t} + V_1\frac{\partial\phi_1}{\partial x}(x,0,t) + g\,\eta(x,t)\right] - T\frac{\partial^2\eta}{\partial x^2}$$

$$= \rho_2\left[\frac{\partial\phi_2}{\partial t} + V_2\frac{\partial\phi_2}{\partial x}(x,0,t) + g\,\eta(x,t)\right]$$

BERNOULLI'S EQUATION FOR $\omega = 0$

LOOK FOR SOLUTION OF FORM

$$\phi_1(x,y,t) = F_1(y) \exp\left[i(kx-\omega t) \right]$$

$$\phi_2(x,y,t) = F_2(y) \exp\left[i(kx-\omega t) \right]$$

$$\eta(x,t) = \eta_0 \exp\left[i(kx-\omega t) \right]$$

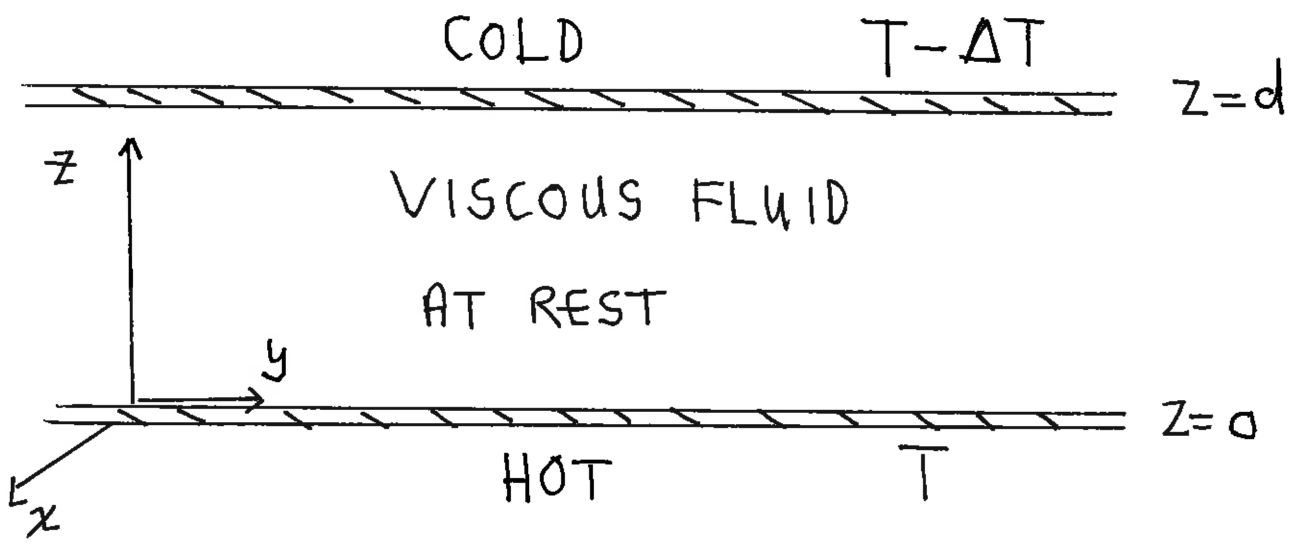
DERIVE DISPERSION RELATION

$$\omega = \omega(k)$$

INVESTIGATE STABILITY OF FLOW WHEN

- $\rho_1 \neq \rho_2$ AND $T \neq 0$
- $\rho_1 = \rho_2$ AND $T \neq 0$
- $\rho_1 \neq \rho_2$ AND $T = 0$

5. THERMAL CONVECTION: BENARD PROBLEM



STATE OF REST STABLE UNTIL ΔT REACHES CRITICAL VALUE.

LINEAR STABILITY THEORY

BOUSSINESQ APPROXIMATION:

ρ CONSTANT EXCEPT IN BODY FORCE TERM IN NAVIER-STOKES EQUATION

$$\rho = \bar{\rho}\left[1 - \alpha(T - \bar{T})\right] \qquad \alpha = \text{COEFFICIENT OF THERMAL EXPANSION}$$

CONSERVATION OF MASS: $\underline{\nabla} \cdot \underline{V} = 0$

MOMENTUM EQUATION: $\rho \frac{D\underline{V}}{Dt} = -\underline{\nabla} p + \mu \nabla^2 \underline{V} + \rho \underline{g}$

ENERGY EQUATION: $\frac{DT}{Dt} = \kappa \nabla^2 T$

EQUATION OF STATE: $\rho = \bar{\rho}\left[1 - \alpha(T - \bar{T})\right]$

UNDISTURBED STATE:

$$0 = -\frac{dp_0}{dz} - \rho_0(z) g$$

$$0 = \kappa \frac{d^2 T_0}{dz^2}$$

$$\rho_0(z) = \bar{\rho}\left[1 - \alpha\left(T_0(z) - \bar{T}\right)\right]$$

PERTURBATION

$$T = T_0(z) + T_1(x,y,z,t), \qquad \rho = \rho_0(z) + \rho_1(x,y,z,t),$$

$$p = p_0(z) + p_1(x,y,z,t), \qquad \underline{V} = \underline{0} + V_1(x,y,z,t).$$

LINEARISED PERTURBATION EQUATIONS

$$\rho_1 = -\alpha \bar{\rho}\, T_1$$

$$\nabla \cdot \underline{V}_1 = 0$$

$$\rho_0 \frac{\partial \underline{V}_1}{\partial t} = -\underline{\nabla} p_1 + \mu \nabla^2 \underline{V}_1 + \rho_1 g$$

$$\frac{\partial T_1}{\partial t} + W_1 \frac{dT_0}{dz} = \kappa \nabla^2 T_1 .$$

- DERIVE ONE EQUATION FOR VELOCITY COMPONENT w_1
- HOMOGENEOUS BOUNDARY CONDITIONS
- EIGENVALUE PROBLEM
- SHOW UNSTABLE IF

$$\mathcal{R} > \frac{27\pi^4}{4}$$

$\mathcal{R}$ = RAYLEIGH NUMBER $= \dfrac{\alpha g d^3 \Delta T}{\nu \kappa}$.